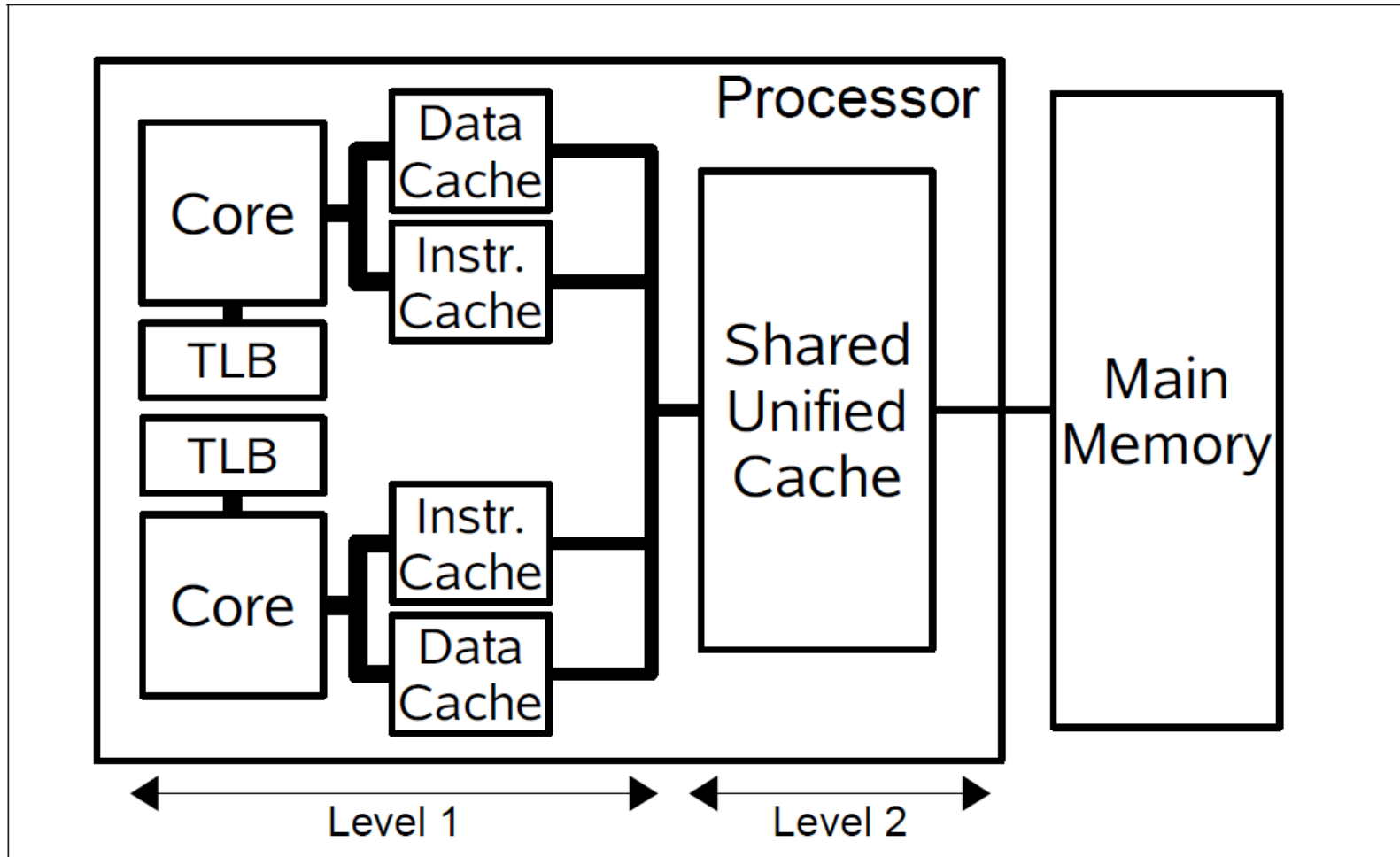


Computational complexity of multithreaded algorithms

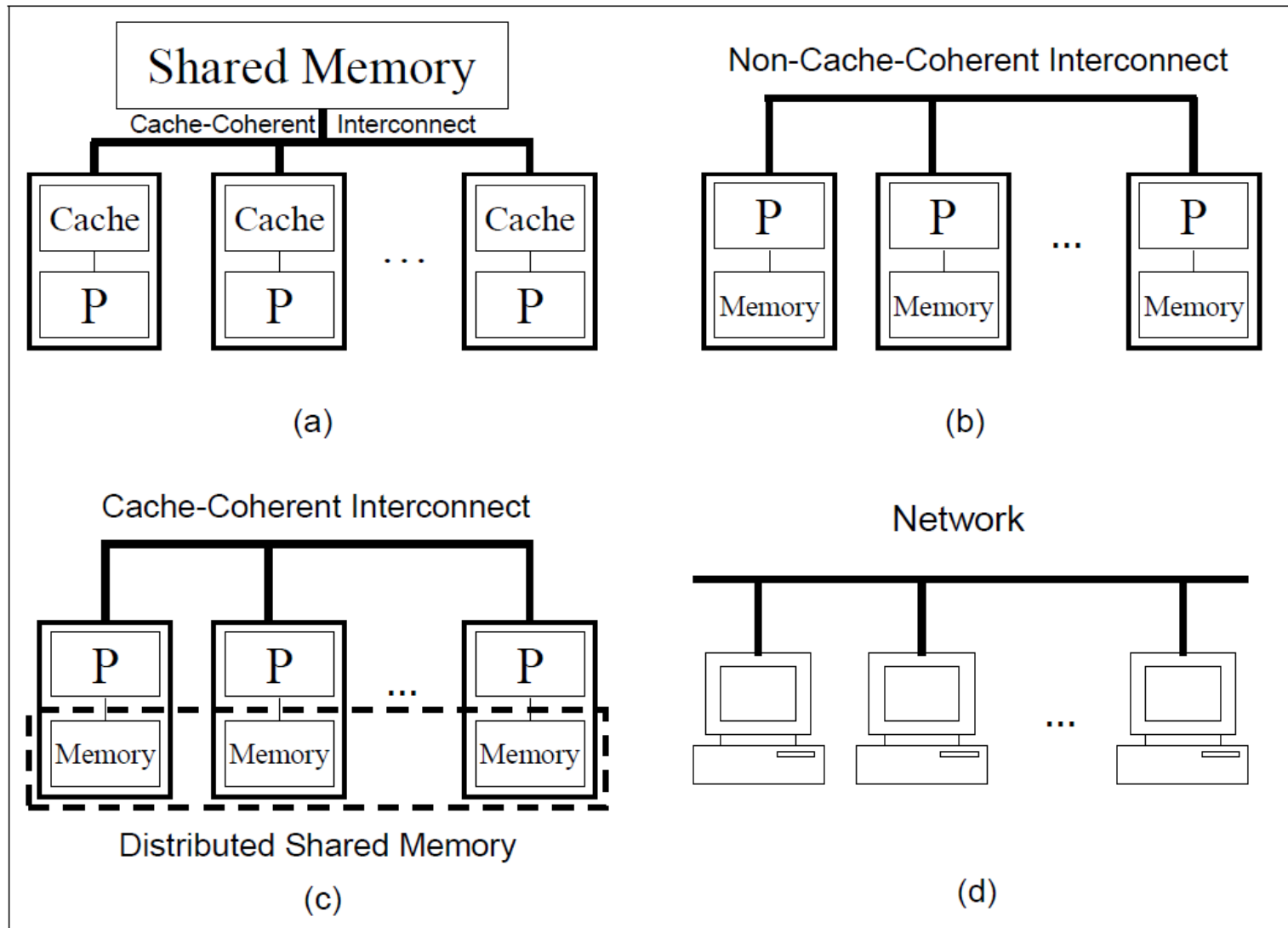


Prof Dr Marko Robnik-Šikonja
Analysis of Algorithms and Heuristic Problem Solving
Edition 2026

Architecture of multi-core processor



Shared and distributed memory



Problems with parallel execution

- deadlock: each member of a group is waiting for another member, including itself, to take action
- happens under the following Coffman conditions
 - mutual exclusion: at least one resource is held in a non-shareable mode (e.g., entering a critical section)
 - hold and wait (resource holding): a process is holding at least one resource and requesting additional resources which are being held by other processes.
 - no preemption: a resource can be released only voluntarily by the process holding it.
 - circular wait: each process must be waiting for a resource which is being held by another process, which in turn is waiting for the first process to release the resource
- livelock: two or more processes continually repeat the same interaction in response to changes in the other processes without doing any useful work.
- starvation: some resource may always be allocated to some process
- race conditions and synchronization: system attempts to perform two or more operations at the same time, but the operations must be done in the proper sequence to be done correctly.

Race conditions

- deterministic and nondeterministic multithreaded programs

```
void Race() {  
    int x = 0 ;  
    parallel for i=1 to 2  
        x = x +1 ;  
    print x ;  
}
```

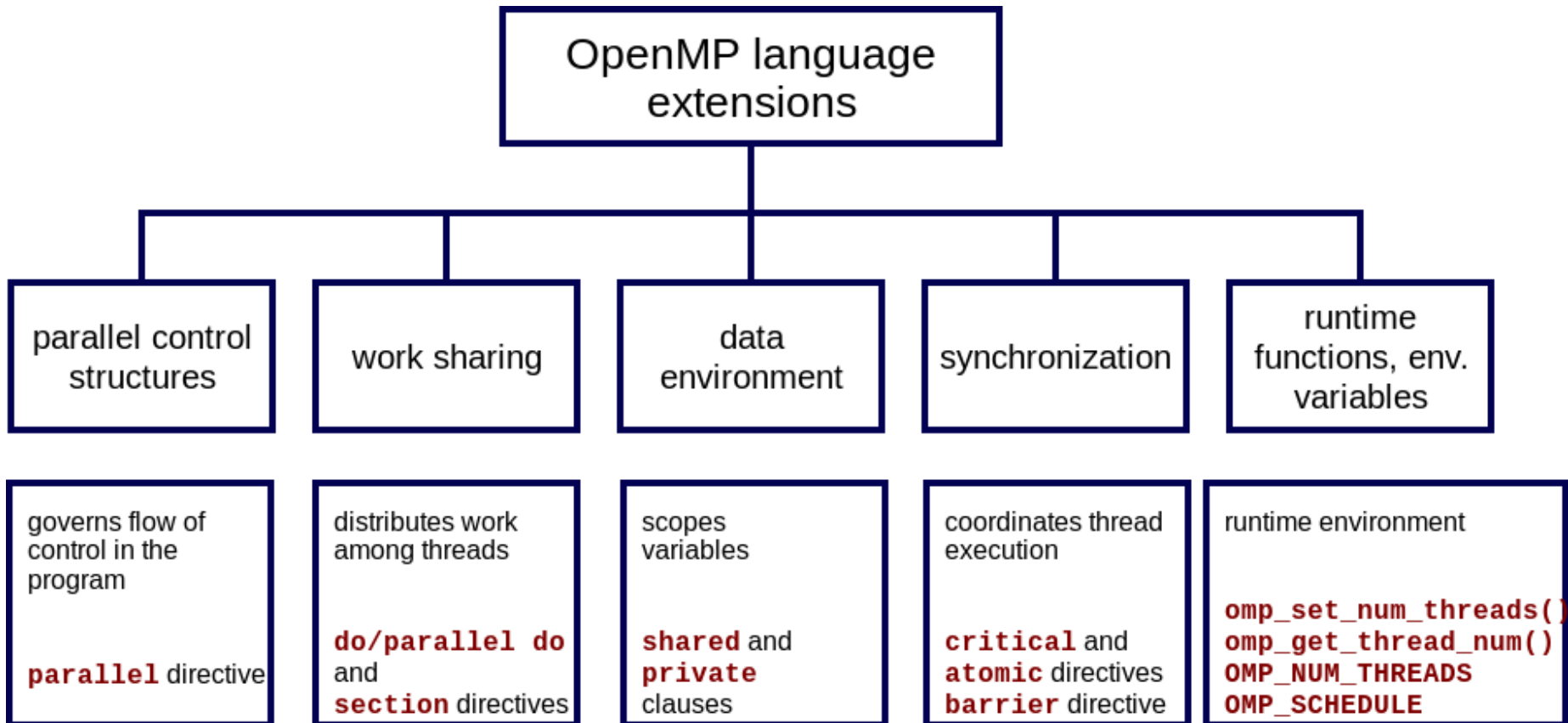
Low level synchronization mechanisms

- monitor: a mechanism that allows threads to have both mutual exclusion and the ability to wait (block) for a certain condition to become false. Monitors also have a mechanism for signaling other threads that their condition has been met.
- semaphore: a (counting) variable controlling access to a common resource
- atomic operations: program operations that cannot be preempted

Dynamic threads

- simplified programming,
- top-level parallelism
- three new constructs: parallel, spawn, sync
- simplified complexity analysis
- platforms: Cilk, Cilk++, OpenMP, Task Parallel Library (.NET), Threading Building Blocks(C++, Intel), JOMP, JPPF (Java)

OMP elements



Example: Fibonacci numbers

$$F_0 = 0$$

$$F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2} \text{ za } n \geq 2$$

- $T(n) = T(n-1) + T(n-2) + \Theta(1)$

- solution $T(n) = \Theta(\tau^n)$
 $\tau = (1 + \sqrt{5})/2$

```
int fib (int n) {  
    if (n <= 1)  
        return n ;  
    else {  
        int x = fib(n-1) ;  
        int y = fib(n-2) ;  
        return x + y ;  
    }  
}
```

Multithreaded Fibonacci

```
int pFib (int n) {  
    if (n <= 1)  
        return n ;  
    else {  
        int x = spawn pFib(n-1) ;  
        int y = pFib(n-2) ;  
        sync  
        return x + y ;  
    }  
}
```

- nested parallelism
- scheduler

Fibonacci with OpenMP

```
int pFib (int n){
    if (n <= 1)
        return n ;
    else {
        int x, y ;
        # pragma omp sections public(x, y)
        {
            #pragma omp section
            x = pFib(n-1) ;

            #pragma omp section
            y = pFib(n-2) ;
        }
        return x + y ;
    }
}
```

Loop parallelization

- Example: multiplication of a matrix and vector $y = A x$

```
void mat_vec(matrix A, vector x) {  
    int n = A.rows ;  
    // let the length of y be n  
    for i = 1 to n  
         $y_i = 0$  ;  
    for i = 1 to n  
        for j = 1 to n  
             $y_i = y_i + A_{ij} * x_j$  ;  
    return y ;  
}
```

```
void mat_vec(matrix A, vector x) {  
    int n = A.rows ;  
    // let the length of y be n  
    parallel for i = 1 to n  
         $y_i = 0$  ;  
    parallel for i = 1 to n  
        for j = 1 to n  
             $y_i = y_i + A_{ij} * x_j$  ;  
    return y ;  
}
```

high-level parallelization

Loop parallelization: actual schedule

// the code, a compiler would generate for the main loop

```
void mat_vec_main_loop(matrix A, vector x, vector y, n, i, k) {  
    if (i == k) {  
        for j = 1 to n  
             $y_i = y_i + A_{ij} * x_j$  ;  
        }  
    else {  
        mid = (i + k) / 2 ; // the floor  
        spawn mat_vec_main_loop(A, x, y, n, i, mid)  
        mat_vec_main_loop(A, x, y, n, mid+1, k)  
        sync  
    }  
}
```

The compiler might generate more coarse-grained parallelization

Multithreaded computational model used in computational complexity analysis

- acyclic directed graph
- equal processors
- no resources for scheduling
- total time, time of parallel tasks
- critical path
- number of processors P , time for P processors T_p
- T_1 , T_∞

Parallel speedup

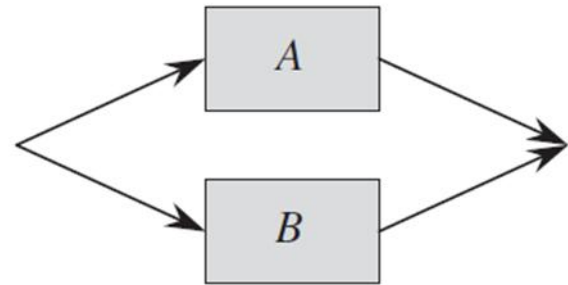
- in one step, using P processors, we finish P units of work, in time T_p we do $P \cdot T_p$ units
- total work is T_1 , note that $P \cdot T_p \geq T_1$
- speedup rule: $T_p \geq T_1 / P$
- also $T_p \geq T_\infty$
- speedup or level of parallelism is $T_1 / T_p \leq P$
- linear speedup $T_1 / T_p = \Theta(P)$
- ideal linear speedup $T_1 / T_p = P$

Analysis of parallel algorithms



Work: $T_1(A \cup B) = T_1(A) + T_1(B)$

Span: $T_\infty(A \cup B) = T_\infty(A) + T_\infty(B)$



Work: $T_1(A \cup B) = T_1(A) + T_1(B)$

Span: $T_\infty(A \cup B) = \max(T_\infty(A), T_\infty(B))$

Analysis of parallel algorithms

- Fibonacci

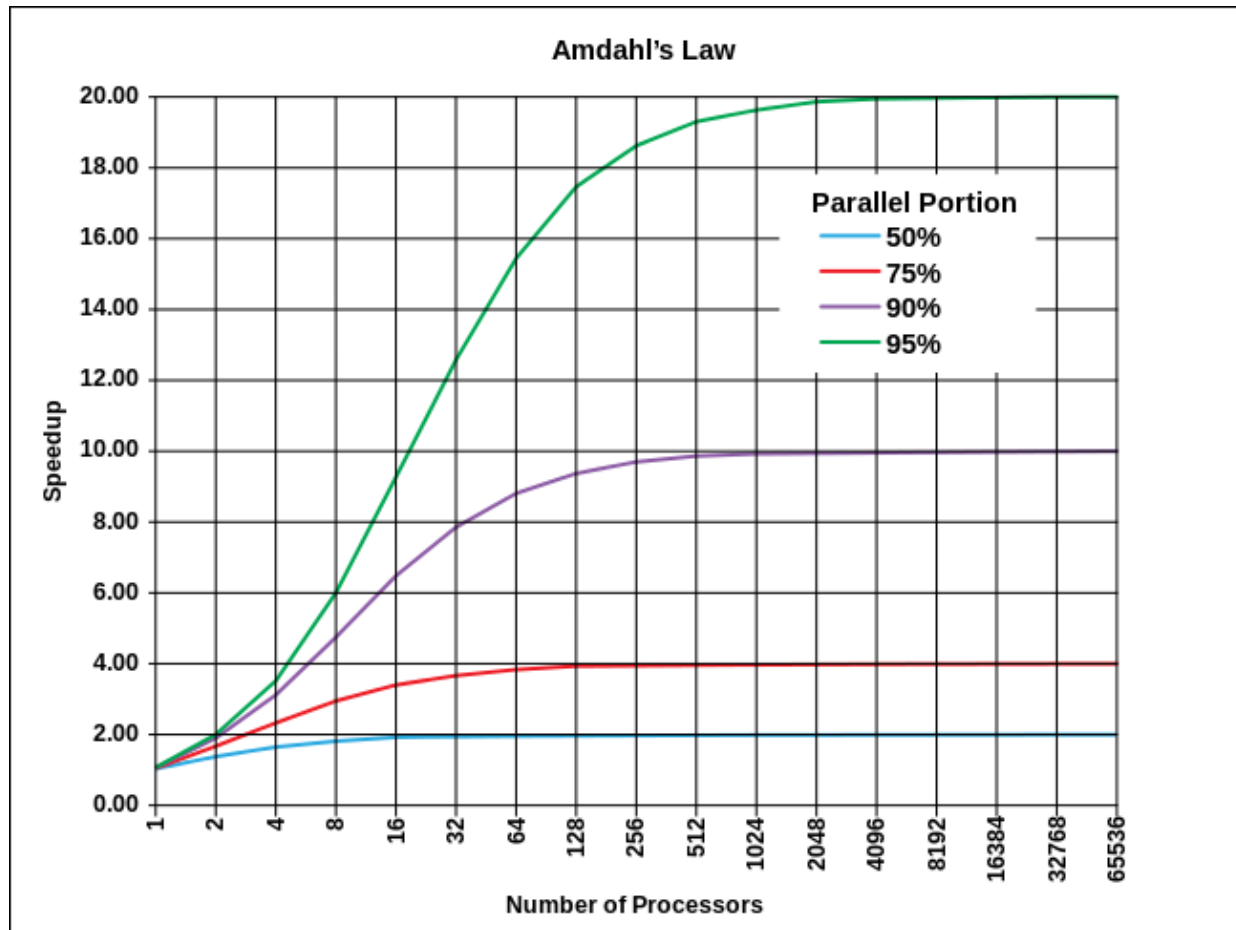
- $T_1(n) = T_1(n-1) + T_1(n-2) + \Theta(1)$

- $T_\infty(n) = \max(T_\infty(n-1) , T_\infty(n-2)) + \Theta(1)$
 $= T_\infty(n-1) + \Theta(1)$
 $= \Theta(n)$

Limits of parallelization

- Amdahl's law
- speedup $S = T_1 / T_p$
- f = proportion of parallelizable code
- $$S = \frac{1}{\frac{f}{P} + (1-f)}$$
- let us compute speedup for 2, 5, 10, ∞ , processors and $f=0.9, 0.5, 0.1$

Amdahl's law

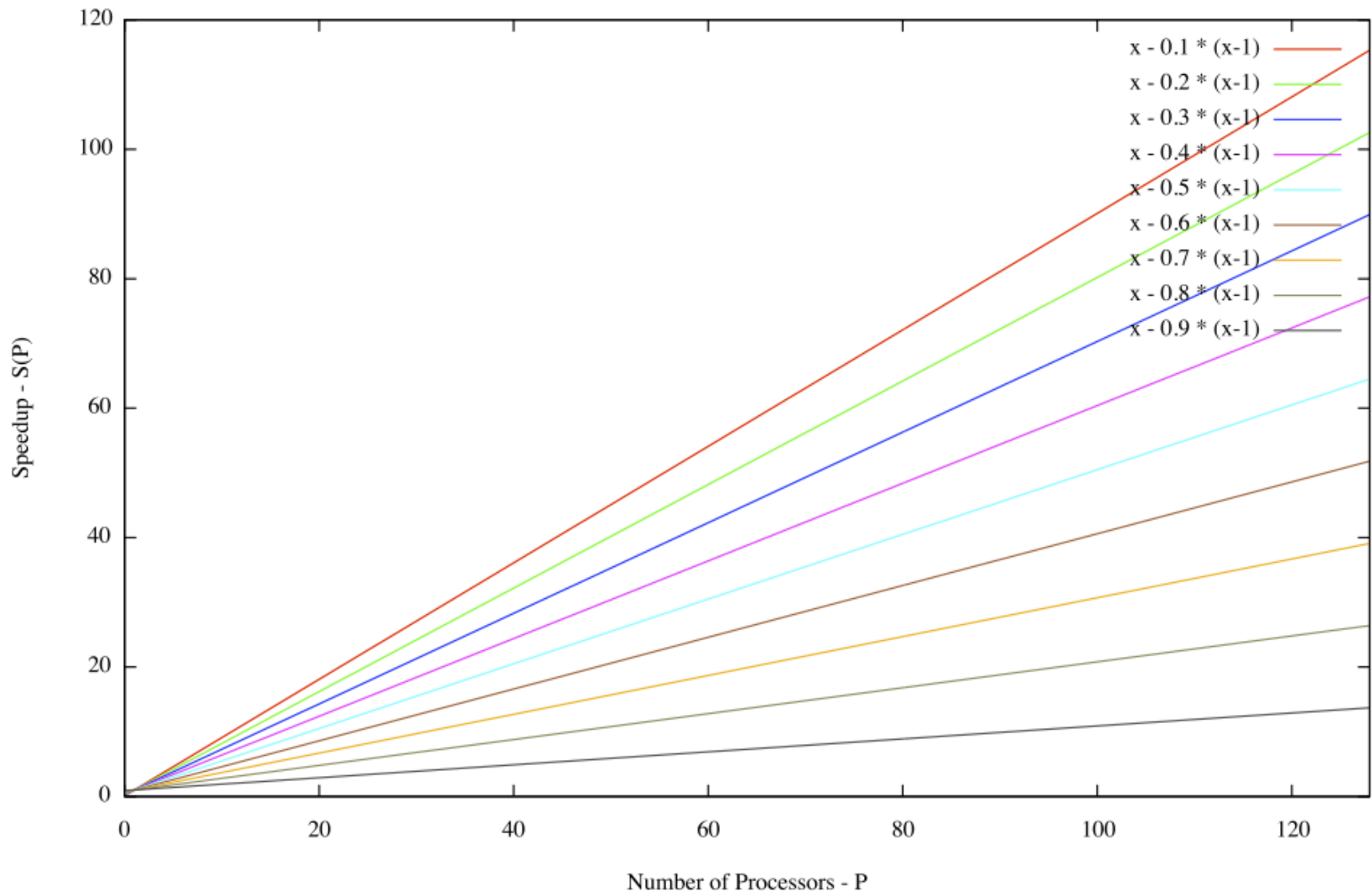


Gustafson's law

- processing time (on each processor) is split to
 $T_1 = a + b$ (a is sequential time, b = parallel time)
Sequential share of work $\alpha = a/(a+b)$
 $1 - \alpha$ is a share of parallel work
- assumption: using more parallel units, we can solve larger problems (or more problems in the same time), the size of problems grows linearly with P , therefore $T_1 = a + P \cdot b$
- speedup $S_p = (a + P \cdot b)/(a+b) = \alpha + P(1 - \alpha) = P - \alpha (P - 1)$
- for small α the speedup is almost linear in P

Speedup by Gustafson

Gustafson's Law: $S(P) = P - a * (P - 1)$



Two views of parallel speedup, Amdahl and Gustafson

- Amdahl: if we travel to a destination 100km away and we used 1 hour for one half of the distance, the total average time will never reach 100km/h, no matter how fast we travel the second half
- Gustafson: suppose you travel for some time with a speed lower than 100km/h; if the distance is long enough and there is enough time available, you can still reach arbitrary average speed; e.g., if you travel 1 hour with the speed 50km/h and continue the next hour with 150km/h, the total average speed will be 100km/h (or you can travel next half an hour with 200km/h)