

# Problem Set 1

Please read the homework submission policies at <https://cs246.stanford.edu>.

## 1 Spark (25 pts)

Write a Spark program that implements a simple “People You Might Know” social network friendship recommendation algorithm. The key idea is that if two people have a lot of mutual friends, then the system should recommend that they connect with each other.

### Data:

- Associated data file is [soc-LiveJournal1Adj.txt](#) in `q1/data`.
- The file contains the adjacency list and has multiple lines in the following format:

`<User><TAB><Friends>`

Here, `<User>` is a unique integer ID corresponding to a unique user and `<Friends>` is a comma separated list of unique IDs corresponding to the friends of the user with the unique ID `<User>`. Note that the friendships are mutual (i.e., edges are undirected): if  $A$  is friend with  $B$  then  $B$  is also friend with  $A$ . The data provided is consistent with that rule as there is an explicit entry for each side of each edge.

**Algorithm:** Let us use a simple algorithm such that, for each user  $U$ , the algorithm recommends  $N = 10$  users who are not already friends with  $U$ , but have the most number of mutual friends in common with  $U$ .

### Output:

- The output should contain one line per user in the following format:

`<User><TAB><Recommendations>`

where `<User>` is a unique ID corresponding to a user and `<Recommendations>` is a comma separated list of unique IDs corresponding to the algorithm’s recommendation of people that `<User>` might know, ordered in decreasing number of mutual friends.

- Even if a user has less than 10 second-degree friends, output all of them in decreasing order of the number of mutual friends. If a user has no friends, you can provide an empty list of recommendations. If there are recommended users with the same number of mutual friends, then output those user IDs in numerically ascending order.

**Pipeline sketch:** Please provide a description of how you used Spark to solve this problem. Don’t write more than 3 to 4 sentences for this: we only want a very high-level description of your strategy to tackle this problem.

**Tips:**

- Use Google Colab to use Spark seamlessly, e.g., copy and adapt the setup cells from Colab 0.
- Before submitting a complete application to Spark, you may go line by line, checking the outputs of each step. Command `.take(X)` should be helpful, if you want to check the first X elements in the RDD.
- We recommend using either PySpark DataFrame and/or RDD syntax for this question.
- For sanity check, your top 10 recommendations for **user ID 11** should be: 27552, 7785, 27573, 27574, 27589, 27590, 27600, 27617, 27620, 27667.

**What to submit**

- (1) Upload your code on Gradescope.
- (2) Include in your writeup a short paragraph sketching your spark pipeline.
- (3) Include in your writeup the recommendations for the users with following user IDs: 924, 8941, 8942, 9019, 9020, 9021, 9022, 9990, 9992, 9993.

★ **Solution** ★

## 2 Association Rules (30 pts)

Association Rules are frequently used for Market Basket Analysis (MBA) by retailers to understand the purchase behavior of their customers. This information can be then used for many different purposes such as cross-selling and up-selling of products, sales promotions, loyalty programs, store design, discount plans and many others.

**Evaluation of item sets:** Once you have found the frequent itemsets of a dataset, you need to choose a subset of them as your recommendations. Commonly used metrics for measuring significance and interest for selecting rules for recommendations are:

1. **Confidence** (denoted as  $\text{conf}(A \rightarrow B)$ ): *Confidence* is defined as the probability of occurrence of  $B$  in the basket if the basket already contains  $A$ :

$$\text{conf}(A \rightarrow B) = \Pr(B|A),$$

where  $\Pr(B|A)$  is the conditional probability of finding item set  $B$  given that item set  $A$  is present.

2. **Lift** (denoted as  $\text{lift}(A \rightarrow B)$ ): *Lift* measures how much more “ $A$  and  $B$  occur together” than “what would be expected if  $A$  and  $B$  were statistically independent”:

$$\text{lift}(A \rightarrow B) = \frac{\text{conf}(A \rightarrow B)}{S(B)},$$

where  $S(B) = \frac{\text{Support}(B)}{N}$  and  $N$  = total number of transactions (baskets).

3. **Conviction** (denoted as  $\text{conv}(A \rightarrow B)$ ): *Conviction* compares the “probability that  $A$  appears without  $B$  if they were independent” with the “actual frequency of the appearance of  $A$  without  $B$ ”:

$$\text{conv}(A \rightarrow B) = \frac{1 - S(B)}{1 - \text{conf}(A \rightarrow B)}.$$

(a) [3pts]

A drawback of using *confidence* is that it ignores  $\Pr(B)$ . Why is this a drawback? Explain why *lift* and *conviction* do not suffer from this drawback.

★ **Solution** ★

(b) [3pts]

A measure is *symmetrical* if  $\text{measure}(A \rightarrow B) = \text{measure}(B \rightarrow A)$ . Which of the measures presented here are symmetrical? For each measure, please provide either a proof that the measure is symmetrical, or a counterexample that shows the measure is not symmetrical.

## ★ Solution ★

(c) [4pts]

*Perfect implications* are rules that hold 100% of the time (or equivalently, the associated conditional probability is 1). A measure is *desirable* if it reaches its maximum achievable value for all perfect implications. This makes it easy to identify the best rules. Which of the above measures have this property? You may ignore 0/0 but not other infinity cases. Also you may find it easy to explain by an example.

## ★ Solution ★

**Application in product recommendations:** The action or practice of selling additional products or services to existing customers is called *cross-selling*. Giving product recommendation is one of the examples of cross-selling that are frequently used by online retailers. One simple method to give product recommendations is to recommend products that are frequently browsed together by the customers.

Suppose we want to recommend new products to the customer based on the products they have already browsed online. Write a program using the *A-priori* algorithm to find products which are frequently browsed together. Fix the support to  $s = 100$  (*i.e.* product pairs need to occur together at least 100 times to be considered frequent) and find itemsets of size 2 and 3.

Use the online browsing behavior dataset from [browsing.txt](#) in q2/data. Each line represents a browsing session of a customer. On each line, each string of 8 characters represents the ID of an item browsed during that session. The items are separated by spaces.

Note: for parts (d) and (e), the writeup will require a specific rule ordering but the program need not sort the output. We are not giving partial credits to coding when results are wrong. However, two sanity checks are provided and they should be helpful when you progress: (1) there are 647 frequent items after 1<sup>st</sup> pass ( $|L_1| = 647$ ), (2) the top 5 pairs you should produce in part (d) all have confidence scores greater than 0.985. See detailed instructions below. Please provide at least five decimal points for your confidence scores.

(d) [10pts]

Identify pairs of items  $(X, Y)$  such that the support of  $\{X, Y\}$  is at least 100. For all such pairs, compute the *confidence* scores of the corresponding association rules:  $X \Rightarrow Y, Y \Rightarrow X$ . Sort the rules in decreasing order of *confidence* scores and list the top 5 rules in the writeup. Break ties, if any, by lexicographically increasing order on the left hand side of the rule. (You need not use Spark for parts (d) and (e) of question 2)

**★ Solution ★****(e) [10pts]**

Identify item triples  $(X, Y, Z)$  such that the support of  $\{X, Y, Z\}$  is at least 100. For all such triples, compute the *confidence* scores of the corresponding association rules:  $(X, Y) \Rightarrow Z$ ,  $(X, Z) \Rightarrow Y$ ,  $(Y, Z) \Rightarrow X$ . Sort the rules in decreasing order of *confidence* scores and list the top 5 rules in the writeup. Order the left-hand-side pair lexicographically and break ties, if any, by lexicographical order of the first then the second item in the pair.

**★ Solution ★****What to submit**

Upload all the code on Gradescope and include the following in your writeup:

- (i) Explanation for 2(a).
- (ii) Proofs and/or counterexamples for 2(b).
- (iii) Explanation for 2(c).
- (iv) Top 5 rules with confidence scores [2(d)].
- (v) Top 5 rules with confidence scores [2(e)].

### 3 Locality-Sensitive Hashing (15 pts)

When simulating a random permutation of rows, as described in **Sec. 3.3.5** of MMDS, we could save time if we restricted our attention to a randomly chosen  $k$  of the  $n$  rows, rather than hashing all  $n$  row numbers. The downside of doing so is that, if none of the  $k$  rows contains a 1 in a certain column, then the result of the minhashing is “don’t know”. In other words, we get no row number as the minhash value. It would be a mistake to assume that two columns that both minhash to “don’t know” are likely to be similar. However, if the probability of getting “don’t know” as a minhash value is small, we can tolerate the situation and simply ignore such minhash values when computing the fraction of minhashes in which two columns agree.

In part (a) we determine an upper bound on the probability of getting “don’t know” as the minhash value when considering only a  $k$ -subset of the  $n$  rows, and in part (b) we use this bound to determine an appropriate choice for  $k$ , given our tolerance for this probability.

(a) [5pts]

Suppose a column has  $m$  1’s and therefore  $n - m$  0’s. We uniformly at random choose a subset of  $k$  distinct rows from the  $n$  rows to consider when computing the minhash. Prove that the probability of getting “don’t know” as the minhash value for this column is at most  $\left(\frac{n-k}{n}\right)^m$ .

★ Solution ★

(b) [5pts]

Suppose we want the probability of “don’t know” to be at most  $e^{-10}$ . Assuming  $n$  and  $m$  are both very large (but  $n$  is much larger than  $m$  or  $k$ ), give a simple approximation to the smallest value of  $k$  that will ensure this probability is at most  $e^{-10}$ . Your expression should be a function of  $n$  and  $m$ . Hints: (1) You may treat  $\left(\frac{n-k}{n}\right)^m$  as the probability of “don’t know” for the purpose of this approximation. (2) Remember that for large  $x$ ,  $(1 - \frac{1}{x})^x \approx 1/e$ .

★ Solution ★

(c) [5pts]

*Note: Part (c) should be considered separate from the previous two parts, in that we are no longer restricting our attention to a randomly chosen subset of the rows.*

When minhashing, one might expect that we could estimate the Jaccard similarity without using all possible permutations of rows. For example, we could only allow cyclic permuta-

tions, i.e. start at a randomly chosen row  $r$ , which becomes the first in the order, followed by rows  $r + 1$ ,  $r + 2$ , and so on, down to the last row, and then continuing with the first row, second row, and so on, down to row  $r - 1$ . There are only  $n$  such permutations if there are  $n$  rows. However, these permutations are not sufficient to estimate the Jaccard similarity correctly.

Give an example of two columns such that the probability (over cyclic permutations only) that their minhash values agree is not the same as their Jaccard similarity. In your answer, please provide (a) an example of a matrix with two columns (let the two columns correspond to sets denoted by  $S1$  and  $S2$ ), (b) the Jaccard similarity of  $S1$  and  $S2$ , and (c) the probability that a random cyclic permutation yields the same minhash value for both  $S1$  and  $S2$ .

### ★ Solution ★

## What to submit

Include the following in your writeup:

- (i) Proof for 3(a)
- (ii) Derivation and final answer for 3(b)
- (iii) Example for 3(c) including the three requested items

## 4 LSH for Approximate Near Neighbor Search (30 pts)

In this problem, we study how locality-sensitive hashing (LSH) can be used to solve the *Approximate Near Neighbor* (ANN) search problem.

**Problem Setup.** Let  $\mathcal{A}$  be a dataset of  $n$  points in a metric space with distance function  $d(\cdot, \cdot)$ . Fix a constant  $c > 1$ . The  $(c, \lambda)$ -Approximate Near Neighbor (ANN) problem is defined as follows: given a query point  $z$ , and assuming that there exists at least one point  $x \in \mathcal{A}$  such that  $d(x, z) \leq \lambda$ , return a point  $x' \in \mathcal{A}$  satisfying  $d(x', z) \leq c\lambda$ . Such a point  $x'$  is called a  $(c, \lambda)$ -ANN of  $z$ .

**LSH Family.** Let  $\mathcal{H}$  be a family of hash functions that is  $(\lambda, c\lambda, p_1, p_2)$ -sensitive with respect to the distance measure  $d(\cdot, \cdot)$ ; that is,

- if  $d(x, y) \leq \lambda$ , then  $\Pr[h(x) = h(y)] \geq p_1$ ;
- if  $d(x, y) > c\lambda$ , then  $\Pr[h(x) = h(y)] \leq p_2$ .

**ANN Algorithm Using LSH.** Define<sup>1</sup>

$$\mathcal{G} = \mathcal{H}^k = \{g = (h_1, \dots, h_k) \mid h_i \in \mathcal{H}, 1 \leq i \leq k\},$$

where

$$k = \log_{1/p_2}(n).$$

Consider the following procedure:

1. Select  $L = n^\rho$  independent hash functions  $g_1, \dots, g_L$  uniformly at random from  $\mathcal{G}$ , where  $\rho = \frac{\log(1/p_1)}{\log(1/p_2)}$ .
2. For each  $1 \leq i \leq L$ , hash all data points in  $\mathcal{A}$  as well as the query point  $z$  using  $g_i$ .
3. Let  $W_i$  denote the set of data points that hash to the same value as  $z$  under  $g_i$ . Uniformly at random, retrieve at most  $3L$  data points from

$$\bigcup_{i=1}^L W_i.$$

(If fewer than  $3L$  data points are present, retrieve all of them.)

4. Among the retrieved points, return the one with minimum distance to  $z$  as the output  $(c, \lambda)$ -ANN.

The goal of the following questions is to show that this procedure returns a correct  $(c, \lambda)$ -ANN with constant probability.

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<sup>1</sup>The equality  $\mathcal{G} = \mathcal{H}^k$  means that each function  $g \in \mathcal{G}$  is an AND-construction of  $k$  functions from  $\mathcal{H}$ ; that is,  $g(x) = g(y)$  if and only if  $h_i(x) = h_i(y)$  for all underlying hash functions  $h_i$ .

(a) [5 pts]

For each  $j = 1, \dots, L$ , define

$$W_j = \{x \in \mathcal{A} \mid g_j(x) = g_j(z)\},$$

the set of points that collide with the query point  $z$  under hash function  $g_j$ .

Define the set of *far (bad) points*

$$T = \{x \in \mathcal{A} \mid d(x, z) > c\lambda\}.$$

Prove that

$$\Pr \left[ \sum_{j=1}^L |T \cap W_j| \geq 3L \right] \leq \frac{1}{3}.$$

(Hint: Markov's Inequality.)

★ Solution ★

(b) [5 pts]

Let  $x^* \in \mathcal{A}$  be a point such that  $d(x^*, z) \leq \lambda$ . Prove:

$$\Pr [\forall 1 \leq j \leq L, g_j(x^*) \neq g_j(z)] < \frac{1}{e}.$$

★ Solution ★

(c) [5 pts]

Using the results of parts (a) and (b), conclude that with probability greater than some fixed constant the reported point is an actual  $(c, \lambda)$ -ANN.

★ Solution ★

(d) [15 pts]

You do not need to use results from parts (a)–(c) to complete part (d).

A dataset of images,<sup>2</sup> `patches.csv`, is provided in `q4/data`.

Each row in this dataset is a  $20 \times 20$  image patch represented as a 400-dimensional vector. We will use the  $L_1$  distance metric on  $\mathbb{R}^{400}$  to define similarity of images. We would like to compare the performance of LSH-based approximate near neighbor search with that of linear search.<sup>3</sup> You should use the code provided with the dataset for this task.

The included starter code in `lsh.py` marks all locations where you need to contribute code with `TODOs`. In particular, you will need to use the functions `lsh_setup` and `lsh_search` and implement your own linear search. The default parameters  $L = 10, k = 24$  to `lsh_setup` work for this exercise, but feel free to use other parameter values as long as you explain the reason behind your parameter choice.

- For each of the image patches in rows 100, 200, 300, ..., 1000, find the top 3 near neighbors in the entire dataset<sup>4</sup> (excluding the original patch itself) using both LSH and linear search. Report the average search time for LSH and linear search, defined as the mean wall-clock time per query averaged over these 10 query patches.
- Use the same set of 10 query patches above. Assuming  $\{z_j \mid 1 \leq j \leq 10\}$  to be the set of image patches considered (*i.e.*,  $z_j$  is the image patch in row  $100j$ ),  $\{x_{ij}\}_{i=1}^3$  to be the approximate near neighbors of  $z_j$  found using LSH, and  $\{x_{ij}^*\}_{i=1}^3$  to be the (true) top 3 near neighbors of  $z_j$  found using linear search, compute the following error measure:

$$error = \frac{1}{10} \sum_{j=1}^{10} \frac{\sum_{i=1}^3 d(x_{ij}, z_j)}{\sum_{i=1}^3 d(x_{ij}^*, z_j)}$$

Plot the error value as a function of  $L$  (for  $L = 10, 12, 14, \dots, 20$ , with  $k = 24$ ). Similarly, plot the error value as a function of  $k$  (for  $k = 16, 18, 20, 22, 24$  with  $L = 10$ ). Briefly comment on the two plots (one sentence per plot would be sufficient).

- Finally, plot the top 10 near neighbors found<sup>5</sup> by linear search and LSH-based search (using the default  $L = 10, k = 24$  or your alternative choice of parameter values for LSH) for the image patch in row 100 (*i.e.* index 99 when using zero-indexing), together with the image patch itself. You may find the function `plot` useful. How do they compare visually?

## What to submit

- Include the proof for 4(a) in your writeup.

<sup>2</sup>Dataset and code adopted from Brown University's Greg Shakhnarovich

<sup>3</sup>By linear search we mean comparing the query point  $z$  directly with every database point  $x$ .

<sup>4</sup>Sometimes, the function `lsh_search` may return less than 3 nearest neighbors. You can use a `while` loop to check that `lsh_search` returns enough results, or you can manually run the program multiple times until it returns the correct number of neighbors.

<sup>5</sup>Same remark, you may sometimes have less than 10 nearest neighbors in your results; you can use the same hacks to bypass this problem.

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- (ii) Include the proof for 4(b) in your writeup.
- (iii) Include the reasoning for why the reported point is an actual  $(c, \lambda)$ -ANN in your writeup [4(c)].
- (iv) Include the following in your writeup for 4(d):
- Average search time for LSH and linear search.
  - Plots for error value vs.  $L$  and error value vs.  $K$ , and brief comments for each plot
  - Plot of 10 nearest neighbors found by the two methods (also include the original image) and brief visual comparison
- (v) Upload the code for 4(d) on Gradescope.

★ Solution ★